NATIONAL UNIVERSITY OF SINGAPORE

PC4243: Atomic & Molecular Physics II

(Semester I: AY 20010-11)

Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This exam paper contains **FOUR** questions and comprises **SIX** printed pages.
2. You have to answer **THREE** out of the four questions.
3. This is a CLOSED BOOK examination
4. Please use only the supplied answer books, and don't mix answers to different problems on the same sheet.
5. There is a table of Clebsch-Gordan coefficients attached.

1: Two level atom in a laser field

The Hamiltonian for a two level atom interacting with a laser is given by

$$H = \frac{\hbar}{2}\left(\Delta|\,1\,\rangle\langle\,1\,| - \Delta|\,2\,\rangle\langle\,2\,| + \Omega e^{-i\phi}|\,1\,\rangle\langle\,2\,| + \Omega e^{i\phi}|\,2\,\rangle\langle\,1\,|\right)$$

where Ω can be taken to be real and positive.

(a) Give a brief description of the evolution of the Bloch vector

$$\mathbf{x} = (2\mathrm{Re}(c_2 c_1^*), 2\mathrm{Im}(c_2 c_1^*), |c_1|^2 - |c_2|^2)$$

associated with the state $c_1|\,1\,\rangle + c_2|\,2\,\rangle$.

(b) Suppose the atom starts out in the state $|\,2\,\rangle$ and the laser detuning is swept from $-\Delta_0$ to Δ_0 in a time τ where $\Delta_0 \gg \Omega$, and $\Omega\tau \gg 1$. Describe the evolution of the Bloch vector during the detuning sweep. What is the final state of the atom?

(c) Suppose an atom in the state $|\,2\,\rangle$ is subjected to a pulse of duration τ where $\Omega\tau = \pi$. Give a qualitative description of the probability of finding the atom in state $|\,1\,\rangle$ after the pulse as a function of the detuning Δ. A simple sketch and an explanation of the positions of the first zeros will suffice.

(d) Suppose an atom in the state $|\,2\,\rangle$ is subjected to two pulses of duration τ separated by a time $T \gg \tau$ and $\Omega\tau = \pi/2$. Give a qualitative description of the probability of finding the atom in state $|\,1\,\rangle$ after the second pulse as a function of the detuning Δ. Explain how this Ramsey pulse sequence can be used for a high precision frequency measurement in a two level system.

— Please turn over —

2: Saturated Absorption

In steady state, the excited state population for a two level atom interacting with a laser field is given by

$$\rho_{ee} = \frac{1}{2}\frac{I/I_s}{1 + I/I_s + 4\Delta^2/\Gamma^2}$$

where I_s is the saturation intensity, Δ is the laser detuning from the atomic resonance, and Γ is the linewidth of the excited state.

(a) Explain what is meant by saturation for a two level atom in a laser field.

(b) Explain how saturation affects the percentage absorption as a function of intensity for a laser passing through an atomic vapor cell.

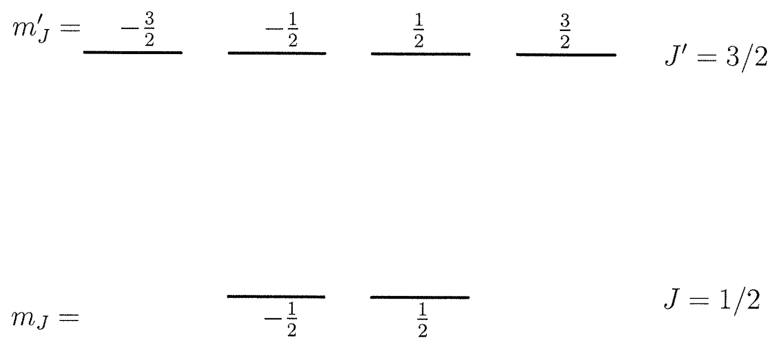
(c) Explain how power broadening arises and derive an expression for the power broadened linewidth for a two level atom.

(d) Give a brief *qualitative* account of the principle of Doppler-free saturated absorption spectroscopy. Your account should explain what determines the heights and widths of the peaks in the transmission spectrum, and the appearance of cross-over resonances in the spectrum of multi-level atoms. No detailed calculations are necessary.

— Please turn over —

3: Laser Cooling

Consider an atom that has a lower level with $J = 1/2$ and an upper level with $J' = 3/2$ as depicted below.



(a) Determine the Clebsch-Gordan coefficients for all electrical dipole-transitions between the two levels.

(b) Suppose that the atom moves through a standing wave with a total electric field

$$\mathbf{E} = |\mathbf{E}_0| \, \mathrm{Re} \left[e^{-i\omega t} \left(\cos(kz)\hat{\mathbf{e}}_+ + i \sin(kz)\hat{\mathbf{e}}_- \right) \right]$$

where $\hat{\mathbf{e}}_\pm$ are the unit vectors for $\sigma^\pm$ polarization. Determine the light shift potentials as a function of position for the $m_J = \pm 1/2$ ground-states in terms of the laser detuning Δ, electric field $E_0 = |\mathbf{E}_0|$, and dipole matrix element

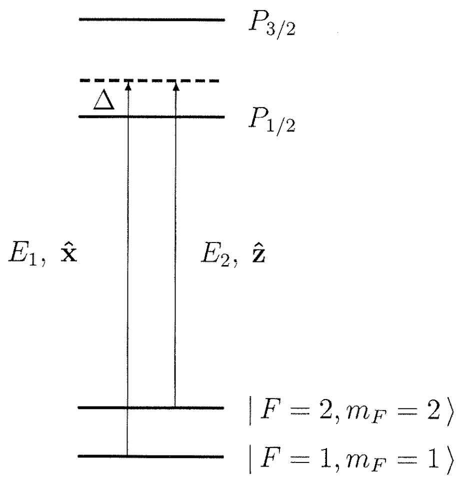
$$\mu = \langle \, J = 1/2, m_J = 1/2 \, | \, e\mathbf{r} \cdot \hat{\mathbf{e}}_+ | \, J' = 3/2, m'_J = 3/2 \, \rangle.$$

(c) Explain the principle of Sisyphus cooling. How does the final temperature scale with the intensity I and detuning Δ of the laser beams? What determines the ultimate limit to the cooling process?

(d) Explain how a magnetic field might influence the sub-doppler cooling process in this system. No detailed calculations are necessary.

— Please turn over —

4: Raman transitions



Consider a ^{87}Rb atom coupled to two laser fields with electric field amplitudes E_1 and E_2, and polarizations $\hat{\mathbf{x}} = (\mathbf{e}_+ + \mathbf{e}_-)/\sqrt{2}$ and $\hat{\mathbf{z}}$ as indicated in the figure above. The detuning Δ from the $\mathrm{P}_{1/2}$ level can be assumed to be much larger than the hyperfine splittings of all levels concerned.

(a) Given that the nuclear spin of ^{87}Rb is $I = 3/2$. Find the expansion of the basis states $|\,F=2, m_F=2\,\rangle$ and $|\,F=1, m_F=1\,\rangle$ in terms of the basis states $|\,J, m_J, I, m_I\,\rangle$.

(b) The two beams induce Raman transitions between the two lower states. If the transition rate is to be independent of the motional state of the atom, along what direction(s) should the beams propagate relative the quantization axis? Determine an expression for the Raman rate Ω_R in this case in terms of the electric field amplitudes E_k, detuning Δ, fine structure splitting ω_F, and dipole matrix element

$$\mu = \langle\, F=2, m_F=2\,|e\mathbf{r}\cdot\mathbf{e}_+|\,J=3/2, m_J=3/2, I=3/2, m_I=3/2\,\rangle.$$

(c) Suppose one wishes to perform Raman side band cooling on an atom confined in a trap. The cooling dimension of interest has a trapping frequency ω_T with the trap axis along $\hat{\mathbf{y}}$. Along what direction(s) would you choose for the beams to propagate relative to the quantization in this case? Explain your choice. What conditions must be satisfied by ω_T and the relevant Raman rate for successful ground state cooling?

[MDB]

— End of paper —

32. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 32.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.